

Coherent States on Hilbert Modules

S. Twareque Ali¹

Department of Mathematics and Statistics, Concordia University
1455 De Maisonneuve Blvd. West, Montréal,
Québec, Canada H3G 1M8

T. Bhattacharyya²

Department of Mathematics, Indian Institute of Science,
Bangalore 560012, India

S. Shyam Roy³

Department of Mathematics and Statistics,
Indian Institute of Science Education and Research, Kolkata
Mohanpur Campus, PO: BCKV Campus Main Office, Mohanpur 741252,
Nadia, West Bengal, India

Abstract

We generalize the concept of coherent states, traditionally defined as special families of vectors on Hilbert spaces, to Hilbert modules. We show that Hilbert modules over C^* -algebras are the natural settings for a generalization of coherent states defined on Hilbert spaces. We consider those Hilbert C^* -modules which have a natural left action from another C^* -algebra say, \mathcal{A} . The coherent states are well defined in this case and they behave well with respect to the left action by \mathcal{A} . Certain classical objects like the Cuntz algebra are related to specific examples of coherent states. Finally we show that coherent states on modules give rise to a completely positive kernel between two C^* -algebras, in complete analogy to the Hilbert space situation. Related to this there is a dilation result for positive operator valued measures, in the sense of Naimark. A number of examples are worked out to illustrate the theory.

¹Supported in part by an NSERC grant

²Supported in part by DST (Ramanna Fellowship) and UGC SAP Phase IV.

³Supported in part by the National Board for Higher Mathematics, India.

I Introduction

Coherent states (CS) are well-known objects in the physical literature. Mathematically, they are an *overcomplete* set of vectors in a Hilbert space, labeled by points in some measure space and satisfying a certain *resolution of the identity* condition. Although coherent states are defined and constructed in a variety of ways, a general construction, and one which will be the prototypical model for the generalization being proposed in this paper, may be described as follows: Let (X, μ) be a finite measure space (often one requires that $\mu(X) = 1$), with X usually being a locally compact space, representing the physical phase space of a classical mechanical system, the homogenous space associated to some physical symmetry group, a coadjoint orbit of a locally compact group, etc. Consider the Hilbert space $L^2(X, d\mu)$ and let Φ_k , $k = 0, 1, 2, 3, \dots, N$ be an *orthonormal* set of vectors (N is generally infinite, but it could also be finite) in it, which satisfy the following condition,

$$\mathcal{N}(x) := \sum_{k=0}^{\infty} |\Phi_k(x)|^2 < \infty, \quad x \in X. \quad (1.1)$$

Coherent states are now defined, for each $x \in X$ to be the vectors,

$$|x\rangle = \sum_{k=0}^N \Phi_k \overline{\Phi_k(x)} \in L^2(X, d\mu), \quad (1.2)$$

Let \mathfrak{H}_K denote the subspace of $L^2(X, d\mu)$ spanned by the Φ_k . Then, the following *resolution of the identity* is easily established.

$$\int_X |x\rangle \langle x| d\mu(x) = I_{\mathfrak{H}_K}. \quad (1.3)$$

It is also easily checked that \mathfrak{H}_K is a *reproducing kernel Hilbert space*, with reproducing kernel $K(x, y) = \langle x | y \rangle$ (see Section III below). A slight variant of this construction is also used. Let \mathfrak{R} be another Hilbert space of dimension N and $\{\psi_k\}_{k=0}^N$ an orthonormal basis of it. With the same vectors Φ_k as before, we alternatively define coherent states as

$$|x\rangle = \sum_{k=0}^N \psi_k \overline{\Phi_k(x)} \in \mathfrak{R}, \quad (1.4)$$

which again satisfy a similar resolution of the identity on \mathfrak{R} .

This rather simple construction of an *overcomplete* family of vectors in a Hilbert space, satisfying a resolution of the identity, turns out to be a powerful tool in many areas of physics and mathematics. Detailed expositions of the theory of coherent states and their applications to mathematics and physics may be found in [3, 11, 14].

The purpose of the present paper is to suggest a possible construction of similar over-complete families of vectors in Hilbert C^* -modules (loosely speaking, Hilbert spaces over C^* -algebras). We shall call the resulting vectors *module valued coherent states (MVCS)*. It is clear that since the field of complex numbers \mathbb{C} is trivially a C^* -algebra, coherent states on Hilbert spaces are special cases of MVCS. The richness of the present generalization will be displayed with a number of examples. Some definitions and preliminary properties of Hilbert C^* -modules have been collected in Appendix V.1.

II Definition and construction of module valued coherent states

Let \mathbf{E} be an $\mathcal{A} - \mathcal{B}$ correspondence, where \mathcal{A} and \mathcal{B} are unital C^* -algebras. This means that \mathbf{E} is a Banach space which is a Hilbert C^* -module over \mathcal{B} , with a left action from \mathcal{A} , that is, there is a $*$ -homomorphism from \mathcal{A} into $\mathcal{L}(\mathbf{E})$. Note that $\mathcal{L}(\mathbf{E})$ denotes the bounded adjointable operators on \mathbf{E} [13, p. 8]. Note also that \mathbf{E} comes equipped with a \mathcal{B} -valued inner product: $\langle \cdot | \cdot \rangle_{\mathbf{E}} : \mathbf{E} \times \mathbf{E} \longrightarrow \mathcal{B}$ given by $(f, g) \mapsto \langle f | g \rangle_{\mathbf{E}}$ for $f, g \in \mathbf{E}$, which is antilinear in the first variable and linear in the second. Furthermore, \mathbf{E} is complete in the norm $\|f\|_{\mathbf{E}} = [\|\langle f | f \rangle_{\mathbf{E}}\|_{\mathcal{B}}]^{\frac{1}{2}}$. Let (X, μ) be a finite measure space and consider the set of functions,

$$\mathbb{F} = \{F : X \longmapsto \mathbf{E} \mid F \text{ is a measurable function}\}.$$

Then clearly, for any two F, G in \mathbb{F} , $x \longmapsto \langle F(x) | G(x) \rangle_{\mathbf{E}}$ is a measurable function. Let

$$\mathfrak{H} = \{F \in \mathbb{F} \mid \text{the function } \langle F(x) | F(x) \rangle \text{ is Bochner integrable}\}. \quad (2.1)$$

Lemma II.1. *\mathfrak{H} is a complex vector space and an inner product module over \mathcal{B} .*

Proof. That \mathfrak{H} is a complex vector space follows from the fact that a necessary and sufficient condition for $\langle F(x) | F(x) \rangle$ to be Bochner integrable is that $\int_X \|\langle F(x) | F(x) \rangle_{\mathbf{E}}\|_{\mathcal{B}} d\mu(x) < \infty$. Indeed, if $F, G \in \mathfrak{H}$, then

$$\begin{aligned} \langle F(x) + G(x) | F(x) + G(x) \rangle_{\mathbf{E}} &= \langle F(x) | G(x) \rangle_{\mathbf{E}} + \langle G(x) | F(x) \rangle_{\mathbf{E}} \\ &+ \langle F(x) | F(x) \rangle_{\mathbf{E}} + \langle G(x) | G(x) \rangle_{\mathbf{E}}. \end{aligned}$$

But, we also know (see, for example [13]) that,

$$\|\langle G(x) | F(x) \rangle_{\mathbf{E}}\|_{\mathcal{B}} \leq \|\langle G(x) | G(x) \rangle_{\mathbf{E}}\|_{\mathcal{B}}^{\frac{1}{2}} \|\langle F(x) | F(x) \rangle_{\mathbf{E}}\|_{\mathcal{B}}^{\frac{1}{2}}.$$

By Schwarz inequality,

$$\begin{aligned} \int_X \|\langle G(x) \mid F(x) \rangle_{\mathbf{E}}\|_{\mathcal{B}} d\mu(x) &\leq \int_X \|\langle G(x) \mid G(x) \rangle_{\mathbf{E}}\|_{\mathcal{B}}^{\frac{1}{2}} \|\langle F(x) \mid F(x) \rangle_{\mathbf{E}}\|_{\mathcal{B}}^{\frac{1}{2}} d\mu(x) \\ &\leq \left(\int_X \|\langle G(x) \mid G(x) \rangle_{\mathbf{E}}\|_{\mathcal{B}} d\mu(x) \right) \\ &\times \left(\int_X \|\langle F(x) \mid F(x) \rangle_{\mathbf{E}}\|_{\mathcal{B}} d\mu(x) \right) < \infty . \end{aligned}$$

Similarly,

$$\int_X \|\langle F(x) \mid G(x) \rangle_{\mathbf{E}}\|_{\mathcal{B}} d\mu(x) < \infty ,$$

so that

$$\int_X \|\langle F(x) + G(x) \mid F(x) + G(x) \rangle_{\mathbf{E}}\|_{\mathcal{B}} d\mu(x) < \infty .$$

In other words $F + G \in \mathfrak{H}$. It is easy to see that \mathfrak{H} is closed under multiplication by complex scalars.

To make \mathfrak{H} an inner product module over \mathcal{B} , we define the right multiplication and the inner product respectively by

$$(F \cdot b)(x) = F(x)b \text{ for all } b \in \mathcal{B}, \quad \langle F \mid G \rangle_{\mathfrak{H}} = \int_X \langle F(x) \mid G(x) \rangle_{\mathbf{E}} d\mu(x)$$

on it. Then, for $b \in \mathcal{B}$,

$$\begin{aligned} \langle F \mid G \cdot b \rangle_{\mathfrak{H}} &= \int_X \langle F(x) \mid G(x)b \rangle_{\mathbf{E}} d\mu(x) \\ &= \int_X \langle F(x) \mid G(x) \rangle_{\mathbf{E}} b d\mu(x) \\ &= \int_X \langle F(x) \mid G(x) \rangle_{\mathbf{E}} d\mu(x) b = \langle F \mid G \rangle_{\mathfrak{H}} b . \end{aligned}$$

■

We have shown that \mathfrak{H} is an inner product \mathcal{B} -module, with respect to the inner product and norm,

$$\langle F \mid G \rangle_{\mathfrak{H}} = \int_X \langle F(x) \mid G(x) \rangle_{\mathbf{E}} d\mu(x) \quad \text{and} \quad \|F\|_{\mathfrak{H}} = \|\langle F \mid F \rangle_{\mathfrak{H}}\|_{\mathcal{B}}^{\frac{1}{2}} .$$

Whenever there is an inner product \mathcal{B} -module, there are certain standard results which follow. We collect these in the following lemma. The proofs can be found, for example, in [13].

Lemma II.2 *For $F, G \in \mathfrak{H}$ we have*

1. $\langle G \mid F \rangle_{\mathfrak{H}} \langle F \mid G \rangle_{\mathfrak{H}} \leq \langle G \mid G \rangle_{\mathfrak{H}} \|\langle F \mid F \rangle_{\mathfrak{H}}\|_{\mathcal{B}}.$
2. $\|\langle F \mid G \rangle_{\mathfrak{H}}\|_{\mathcal{B}} \leq \|F\|_{\mathfrak{H}} \|G\|_{\mathfrak{H}}.$
3. $\|F + G\|_{\mathfrak{H}} \leq \|F\|_{\mathfrak{H}} + \|G\|_{\mathfrak{H}}.$

Lemma II.3 \mathfrak{H} is complete under the norm:

$$\|F\|_{\mathfrak{H}} = \|\langle F \mid F \rangle_{\mathbf{E}}\|_{\mathcal{B}}^{\frac{1}{2}}.$$

Proof. Let $\{F_n\}$ be a Cauchy sequence in \mathfrak{H} . There is a subsequence $\{F_{n_i}\}, n_1 < n_2 < n_3 \dots$, such that

$$\|F_{n_{i+1}} - F_{n_i}\|_{\mathfrak{H}} < 2^{-i} \text{ for } i \in \mathbb{N}. \quad (2.2)$$

Let us observe the following for $G \in \mathfrak{H}$

$$\int_X \langle G(x) \mid G(x) \rangle_{\mathbf{E}} d\mu(x) \geq \int_{\{x: \langle G(x) \mid G(x) \rangle_{\mathbf{E}} \geq \epsilon\}} \langle G(x) \mid G(x) \rangle_{\mathbf{E}} d\mu(x) \quad (2.3)$$

$$\geq \epsilon \mu(\{x : \langle G(x) \mid G(x) \rangle_{\mathbf{E}} \geq \epsilon\}) \quad (2.4)$$

$$(2.5)$$

Hence we have

$$\mu(\{x : \langle G(x) \mid G(x) \rangle_{\mathbf{E}} \geq \epsilon\}) \leq \epsilon^{-1} \int_X \langle G(x) \mid G(x) \rangle_{\mathbf{E}} d\mu(x) \quad (2.6)$$

The Equation (2.2) can be rewritten as

$$\|F_{n_{i+1}} - F_{n_i}\|_{\mathfrak{H}}^2 = \left\| \int_X \langle (F_{n_{i+1}} - F_{n_i})(x) \mid (F_{n_{i+1}} - F_{n_i})(x) \rangle_{\mathbf{E}} d\mu(x) \right\|_{\mathcal{B}} < 2^{-2i} \quad (2.7)$$

Using the fact: if $a \in \mathcal{B}$ is positive and $\|a\| < \delta$, then $a < \delta$, we have

$$\int_X \langle (F_{n_{i+1}} - F_{n_i})(x) \mid (F_{n_{i+1}} - F_{n_i})(x) \rangle_{\mathbf{E}} d\mu(x) < 2^{-2i} \quad (2.8)$$

Putting $A_i = \{x : \langle (F_{n_{i+1}} - F_{n_i})(x) \mid (F_{n_{i+1}} - F_{n_i})(x) \rangle_{\mathbf{E}} \geq 2^{-i}\}$ and applying Equation (2.6) to $(F_{n_{i+1}} - F_{n_i})$ in place of G we obtain from Equation (2.8) that

$$\mu(A_i) \leq 2^i \|F_{n_{i+1}} - F_{n_i}\|_{\mathfrak{H}}^2 < 2^{-i}.$$

Hence we get $\sum_{i=1}^{\infty} \mu(A_i) < \infty$. Now by Borel-Cantelli Lemma we have $\mu(\limsup_n A_n) = 0$, where $\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$. But if $x \notin \limsup_n A_n$, then $\langle (F_{n_{i+1}} - F_{n_i})(x) \mid (F_{n_{i+1}} - F_{n_i})(x) \rangle_{\mathbf{E}} < 2^{-i}$ for large i . Equivalently, $\|\langle (F_{n_{i+1}} - F_{n_i})(x) \mid (F_{n_{i+1}} - F_{n_i})(x) \rangle_{\mathbf{E}}\|_{\mathcal{B}} < 2^{-i}$ for $x \notin \limsup_n A_n$. Hence $\{F_{n_i}(x)\}$ is a Cauchy sequence in \mathbf{E} for $x \notin \limsup_n A_n$. Since \mathbf{E} is

complete in the norm $\|\langle \cdot | \cdot \rangle_{\mathbf{E}}\|_{\mathcal{B}}^{\frac{1}{2}}$ the sequence $\{F_{n_i}\}$ converges pointwise almost everywhere to function F (say). That is,

$$\lim_{i \rightarrow \infty} \|\langle (F_{n_i} - F)(x) | (F_{n_i} - F)(x) \rangle_{\mathbf{E}}\|_{\mathcal{B}} = 0 \text{ for } x \notin \limsup_n A_n. \quad (2.9)$$

We define $F(x) = 0$ for $x \in \limsup_n A_n$. So F is a function on X such that Equation (2.9) holds. If μ is a finite measure then constants are integrable with respect to μ . Hence by Dominated Convergence Theorem we conclude from Equation (2.9) that

$$\lim_{i \rightarrow \infty} \int_X \|\langle (F_{n_i} - F)(x) | (F_{n_i} - F)(x) \rangle_{\mathbf{E}}\|_{\mathcal{B}} d\mu(x) = 0 \quad (2.10)$$

Since

$$\begin{aligned} \|F_{n_i} - F\|_{\mathfrak{H}}^2 &= \left\| \int_X \langle (F_{n_i} - F)(x) | (F_{n_i} - F)(x) \rangle_{\mathbf{E}} d\mu(x) \right\|_{\mathcal{B}} \\ &\leq \int_X \|\langle (F_{n_i} - F)(x) | (F_{n_i} - F)(x) \rangle_{\mathbf{E}}\|_{\mathcal{B}} d\mu(x) \end{aligned}$$

It follows from Equation (2.10) that

$$\lim_{i \rightarrow \infty} \|F_{n_i} - F\|_{\mathfrak{H}} = 0.$$

Since $\{F_n\}$ is a Cauchy sequence we have

$$\lim_{n \rightarrow \infty} \|F_n - F\|_{\mathfrak{H}} = 0.$$

Equation (2.10) allows us to pick some n_i such that

$$\int_X \|\langle (F_{n_i} - F)(x) | (F_{n_i} - F)(x) \rangle_{\mathbf{E}}\|_{\mathcal{B}} d\mu(x) < \infty.$$

Hence $F = (F - F_{n_i}) + F_{n_i} \in \mathfrak{H}$. Therefore \mathfrak{H} is complete in the norm specified. \blacksquare

Note also that \mathfrak{H} is an $\mathcal{A} - \mathcal{B}$ correspondence. This is so because for any $a \in \mathcal{A}$ and $F \in \mathfrak{H}$, we may define $(a \cdot F)(x) = aF(x)$ where, for $f \in \mathbf{E}$, by af we mean the left action of a on f , through its image in $\mathcal{L}(\mathbf{E})$ under the postulated $*$ -homomorphism. Moreover,

$$\begin{aligned} \langle a \cdot F | G \rangle_{\mathfrak{H}} &= \int_X \langle aF(x) | G(x) \rangle_{\mathbf{E}} d\mu(x) \\ &= \int_X \langle F(x) | a^*G(x) \rangle_{\mathbf{E}} d\mu(x), \text{ in view of the left action of } \mathcal{A} \text{ on } \mathbf{E}, \\ &= \langle F | a^* \cdot G \rangle_{\mathfrak{H}}. \end{aligned} \quad (2.11)$$

At this point, let us introduce a notation which we shall use consistently in the sequel. For $e \in \mathbf{E}$, we define the map $\langle e| : \mathbf{E} \longrightarrow \mathcal{B}$, by

$$\langle e|(f) = \langle e | f \rangle_{\mathbf{E}} , \quad f \in \mathbf{E} .$$

This is an *adjointable map*. We shall denote its adjoint by $|e\rangle$. Then $|e\rangle : \mathcal{B} \longrightarrow \mathbf{E}$ has the action

$$|e\rangle(b) = eb , \quad b \in \mathcal{B} ,$$

so that for $e_1, e_2 \in \mathbf{E}$,

$$|e_1\rangle\langle e_2|(f) = e_1\langle e_2 | f \rangle_{\mathbf{E}} . \quad (2.12)$$

Thus formally, one may use the standard “bra-ket” notation for Hilbert modules as one does for Hilbert spaces.

II.1 Non-normalized module valued CS

Proceeding now to construct coherent states, we choose a set of vectors $F_0, F_1, \dots, F_n, \dots$ (finite or infinite) in the function space \mathfrak{H} (see (II)), which are *pointwise defined* (for all $x \in X$) and which satisfy the *orthogonality relations*,

$$\int_X |F_k(x)\rangle\langle F_\ell(x)| d\mu(x) = I_{\mathbf{E}} \delta_{k\ell} . \quad (2.13)$$

Next we take a second Hilbert module \mathbf{G} , over another C^* -algebra \mathcal{C} , which may or may not be the same algebra as \mathcal{B} . In \mathbf{G} we choose a set of elements, $\phi_0, \phi_1, \dots, \phi_n, \dots$, of the same cardinality as of the F_k , and which satisfy,

$$\sum_k |\phi_k\rangle\langle\phi_k| = I_{\mathbf{G}} . \quad (2.14)$$

Note that it follows from (2.14), that any element $f \in \mathbf{G}$ can be written as a linear combination of the ϕ_k , with \mathcal{C} -valued coefficients:

$$f = \sum_k |\phi_k\rangle\langle\phi_k|(f) = \sum_k \phi_k c_k , \quad c_k = \langle\phi_k | f\rangle_{\mathbf{G}} \in \mathcal{C} .$$

Let $\mathbf{H} = \mathbf{E} \otimes \mathbf{G}$ denote the *exterior tensor product* (see, for example, [13]) of the two Hilbert modules \mathbf{E} and \mathbf{G} , which is then itself a Hilbert module over $\mathcal{B} \otimes \mathcal{C}$. Here and elsewhere in the paper, we consider only the projective tensor product of C^* -algebras which is also called special or minimal. In case one of our C^* -algebras is nuclear, all C^* -norms on the algebraic tensor product coincide and hence in that case there is a unique tensor product. For each $x \in X$ and *co-isometry* $a \in \mathcal{A}$ (i.e., $aa^* = \text{id}_{\mathcal{A}}$), we define the vectors,

$$|x, a\rangle = \sum_k a F_k(x) \otimes \phi_k \in \mathbf{H} , \quad (2.15)$$

assuming of course that the sum converges. We call these vectors (non-normalized) *module valued coherent states (MVCS)*.

Lemma II.4. *The MVCS in (2.15) satisfy the resolution of the identity,*

$$\int_X |x, a\rangle\langle x, a| d\mu(x) = I_{\mathbf{H}}. \quad (2.16)$$

Proof.

It is enough to prove the identity on elements in \mathbf{H} of the type $h = e \otimes g$, with $e \in \mathbf{E}$ and $g \in \mathbf{G}$. Since these elements form a total set in \mathbf{H} , the lemma will be proved by extending by continuity. Indeed,

$$\begin{aligned} & \left(\int_X |x, a\rangle\langle x, a| d\mu(x) \right) (e \otimes g) \\ &= \int_X |x, a\rangle\langle x, a| e \otimes g|_{\mathbf{H}} d\mu(x) \\ &= \int_X \sum_k (aF_k(x) \otimes \phi_k) \cdot \sum_{\ell} \langle aF_{\ell}(x) \otimes \phi_{\ell} | e \otimes g \rangle_{\mathbf{H}} d\mu(x) \\ &= \sum_{k, \ell} \int_X (aF_k(x) \otimes \phi_k) (\langle aF_{\ell}(x) | e \rangle_{\mathbf{E}} \otimes \langle \phi_{\ell} | g \rangle_{\mathbf{G}}) d\mu(x) \\ &= \sum_{k, \ell} \int_X aF_k(x) \langle aF_{\ell}(x) | e \rangle_{\mathbf{E}} \otimes \phi_k \langle \phi_{\ell} | g \rangle_{\mathbf{G}} d\mu(x) \\ &= \left(\sum_{k, \ell} \int_X |aF_k(x)\rangle\langle aF_{\ell}(x)|(e) d\mu(x) \right) \otimes \phi_k \langle \phi_{\ell} | g \rangle_{\mathbf{G}}, \quad \text{by virtue of (2.12)} \\ &= \sum_{k, \ell} a \left(\int_X |F_k(x)\rangle\langle F_{\ell}(x)| d\mu(x) \right) a^* e \otimes |\phi_k\rangle\langle \phi_{\ell}| (g) \\ &= aa^* e \otimes \sum_k |\phi_k\rangle\langle \phi_k|(g), \quad \text{in view of (2.13)} \\ &= e \otimes g, \quad \text{by assumption (2.14).} \end{aligned}$$

This proves the resolution of the identity holds for vectors of the postulated type. The lemma is proved, as stated earlier, by continuity. ■

(Note that in the above expressions, we are using the same notation, \otimes , to denote tensor products between different spaces. However, it is clear from the context which spaces are meant in any given instance.)

II.2 Normalized module valued CS

We now proceed to show how the above construction may be modified to obtain MVCS which are normalized.

In constructing the non-normalized MVCS two constraints, (2.13) and (2.14) were imposed. We now impose two additional conditions, in order to obtain the *normalized* versions of these MVCS. We denote the normalized MVCS by $|\widehat{x, a}\rangle$ and require that

$$\langle \widehat{x, a} | \widehat{x, a} \rangle = \text{id}_{\mathcal{B}} \otimes \text{id}_{\mathcal{C}} . \quad (2.17)$$

In order to achieve this we first require that

$$\langle \phi_k | \phi_\ell \rangle_{\mathbf{G}} = \text{id}_{\mathcal{C}} \delta_{k\ell} , \quad \text{for all } k, \ell \geq 0 . \quad (2.18)$$

Next we define

$$\mathcal{N}(x, a) = \langle x, a | x, a \rangle_{\mathbf{H}} = \sum_k \langle F_k(x) | a^* a F_k(x) \rangle_{\mathbf{E}} \otimes \text{id}_{\mathcal{C}} , \quad (2.19)$$

the second equality following from (2.11) and (2.18). We now require that for each $x \in X$,

$$\mathcal{N}(x, \text{id}_{\mathcal{A}}) = \langle x, \text{id}_{\mathcal{A}} | x, \text{id}_{\mathcal{A}} \rangle_{\mathbf{H}} = \sum_k \langle F_k(x) | F_k(x) \rangle_{\mathbf{E}} \otimes \text{id}_{\mathcal{C}} > 0 , \quad (2.20)$$

in the sense that we require the existence of $\mathcal{N}(x, \text{id}_{\mathcal{C}})$ as a *positive invertible element* in the C^* -algebra $\mathcal{B} \otimes \mathcal{C}$. Additionally, we require that a be an invertible element in \mathcal{A} . In that case,

$$\mathcal{N}(x, a) > \inf_{\lambda \in \sigma(a^*a)} \lambda \mathcal{N}(x, \text{id}_{\mathcal{C}}) > 0 ,$$

$\sigma(a^*a)$ denoting the spectrum of a^*a in \mathcal{C} and hence $\mathcal{N}(x, a)$ is invertible in \mathcal{C} .

Finally, we define the *normalized MVCS* as,

$$|\widehat{x, a}\rangle = |x, a\rangle \mathcal{N}(x, a)^{-\frac{1}{2}} . \quad (2.21)$$

It is then straightforward to verify that these CS satisfy the normalization condition (2.17) and the resolution of the identity,

$$\int_X |\widehat{x, a}\rangle \mathcal{N}(x, a) \langle \widehat{x, a} | d\mu(x) = I_{\mathbf{H}} , \quad \mathbf{H} = \mathbf{E} \otimes \mathbf{G} , \quad (2.22)$$

which should be compared to (2.16).

III Some examples

Let us look at a few examples which illustrate the above construction.

Example 1. *Standard coherent states*

First we show that the usual Hilbert space valued coherent states are contained in our definition. As stated in the Introduction, these coherent states can generically be obtained as follows. We start with the Hilbert space $\mathfrak{H} = L^2(X, \mu)$ and assume that it contains a reproducing kernel subspace, which we denote by \mathfrak{H}_K . This means that there exists an integral kernel, $K : X \times X \rightarrow \mathbb{C}$, which satisfies $\overline{K(x, y)} = K(y, x)$, $K(x, x) > 0$, for all $x, y \in X$, and for any $f \in \mathfrak{H}_K$,

$$f(x) = \int_X K(x, y) f(y) d\mu(y), \quad \text{for all } x \in X.$$

Moreover, if \mathbb{P}_K denotes the projection operator from $L^2(X, \mu)$ to the subspace \mathfrak{H}_K , then $K(x, y)$ is the integral kernel of this operator. If $\Phi_0, \Phi_1, \dots, \Phi_n, \dots$ is any orthonormal basis of \mathfrak{H}_K , then

$$K(x, y) = \sum_k \Phi_k(x) \overline{\Phi_k(y)}. \quad (3.1)$$

Using this fact, one can define *non-normalized* coherent states as

$$|x\rangle := K(\cdot, x) = \sum_k \Phi_k \overline{\Phi_k(x)} \in \mathfrak{H}_K. \quad (3.2)$$

It is then easy to verify that

$$\langle x | y \rangle = K(x, y) \quad \text{and} \quad \int_X |x\rangle \langle x| d\mu(x) = I_{\mathfrak{H}_K}. \quad (3.3)$$

Generally, one can take any other Hilbert space \mathfrak{K} , the dimension of which has the same cardinality as that of \mathfrak{H}_K and define coherent states in \mathfrak{K} as

$$|x\rangle = \sum_k \psi_k \overline{\Phi_k(x)},$$

where $\psi_1, \psi_2, \dots, \psi_n, \dots$ is an orthonormal basis of \mathfrak{K} . These CS satisfy both the conditions in (3.3), with $I_{\mathfrak{H}_K}$ replaced by $I_{\mathfrak{K}}$. Furthermore, since $K(x, x) = \sum_k |\Phi_k(x)|^2 := \mathcal{N}(x) > 0$, normalized CS can be defined as:

$$\widehat{|x\rangle} = \mathcal{N}(x)^{-\frac{1}{2}} |x\rangle,$$

which then satisfy, the conditions,

$$\|\widehat{|x\rangle}\| = 1 \quad \text{and} \quad \int_X \widehat{|x\rangle} \widehat{\langle x|} \mathcal{N}(x) d\mu(x) = I_{\mathfrak{K}}.$$

In order to arrive at these coherent states from our previous construction, we take $\mathcal{A} = \mathcal{B} = \mathcal{C} = \mathbb{C}$, also $\mathbf{E} = \mathbb{C}$, $\mathbf{G} = \mathfrak{K}$ (both considered as Hilbert modules over \mathbb{C} and $F_k(x) = \overline{\Phi_k(x)}$).

Example 2. Vector coherent states

Vector coherent states can generically be constructed as follows. Consider the Hilbert space $\mathfrak{H} = L^2_{\mathbb{C}^N}(X, \mu)$, of \mathbb{C}^N -valued functions on X , with scalar product

$$\langle \mathbf{f} \mid \mathbf{g} \rangle_{\mathfrak{H}} = \int_X \mathbf{f}(x)^\dagger \mathbf{g}(x) d\mu(x) .$$

(In our notation, $\mathbf{f}(x)$ is the column vector with components $f_i(x)$ and $\mathbf{f}(x)^\dagger$ is the row vector $(\overline{f_1(x)}, \overline{f_2(x)}, \dots, \overline{f_N(x)})$.) Suppose there exists a reproducing kernel subspace $\mathfrak{H}_{\mathbf{K}} \subset L^2_{\mathbb{C}^N}(X, \mu)$, with a (matrix valued) kernel $\mathbf{K} : X \times X \longrightarrow \mathcal{M}_N(\mathbb{C})$ (set of all $N \times N$ complex matrices). Denote by $\mathbb{P}_{\mathbf{K}}$ the projection operator from \mathfrak{H} to $\mathfrak{H}_{\mathbf{K}}$ and let $\Phi_0, \Phi_1, \dots, \Phi_n, \dots$, be any orthonormal basis of $\mathfrak{H}_{\mathbf{K}}$. Then,

$$\mathbf{K}(x, y) = \sum_k \Phi_k(x) \Phi_k(y)^\dagger , \quad \mathbf{K}(x, y)^* = \mathbf{K}(y, x) , \quad \forall x, y \in X \quad (3.4)$$

$$(\mathbb{P}_{\mathbf{K}} \mathbf{f})(x) = \int_X \mathbf{K}(x, y) \mathbf{f}(y) d\mu(y) , \quad \mathbf{f} \in \mathfrak{H} . \quad (3.5)$$

Furthermore, for each $x \in X$, $\mathcal{N}(x) := \mathbf{K}(x, x) = \sum_k \Phi_k(x) \Phi_k(x)^\dagger$ is a positive, invertible matrix and

$$\int_X \mathbf{K}(x, z) \mathbf{K}(z, y) d\mu(z) = \mathbf{K}(x, y) .$$

Let $\Phi_k^1(x), \Phi_k^2(x), \dots, \Phi_k^N(x)$ denote the components of the N -vector $\Phi_k(x)$ and let $\{\chi^i\}_{i=1}^N$ be an orthonormal basis of \mathbb{C}^N . Vector coherent states (VCS) are now defined to be the elements in $\mathfrak{H}_{\mathbf{K}}$:

$$|x, i\rangle := \mathbf{K}(\cdot, x) \chi^i = \sum_k \Phi_k \overline{\Phi_k^i(x)} , \quad x \in X, \quad i = 1, 2, \dots, N . \quad (3.6)$$

These satisfy the conditions,

$$\langle x, i \mid y, j \rangle = \mathbf{K}(x, y)_{ij} , \quad \sum_{i=1}^N \int_X |x, i\rangle \langle x, i| d\mu(x) = I_{\mathfrak{H}_{\mathbf{K}}} .$$

This time, “normalized” VCS are defined as,

$$|\widehat{x, i}\rangle = [\text{Tr}(\mathcal{N}(x))]^{-\frac{1}{2}} |x, i\rangle \quad \text{so that,} \quad \sum_{i=1}^N \|\widehat{x, i}\|^2 = 1 ,$$

and

$$\sum_{i=1}^N \int_X |\widehat{x, i}\rangle \langle \widehat{x, i}| \text{Tr}(\mathcal{N}(x)) d\mu(x) = I_{\mathfrak{H}_{\mathbf{K}}} .$$

We now show how these VCS can be associated, and in fact obtained, from a family of MVCS. This will be achieved by extending the space X over which the coherent states are defined. Take the group $SU(N)$ of $N \times N$ unitary matrices with determinant one. Let $d\Omega$ denote its Haar measure (normalized to one). It is known from the general theory of compact groups that for any normalized vector $\mathbf{v} \in \mathbb{C}^N$, one has the relation,

$$\int_{SU(N)} u \mathbf{v} \mathbf{v}^\dagger u^* d\Omega(u) = \frac{1}{N} \mathbb{I}_N. \quad (3.7)$$

Consider now the domain $X \times SU(N)$ and the orthonormal basis $\{\Phi_i\}$ of the reproducing kernel Hilbert space $\mathfrak{H}_{\mathbf{K}}$, considered above. Let us now define the matrix valued functions $F_k : X \times SU(N) \rightarrow \mathcal{M}_N(\mathbb{C})$,

$$\begin{aligned} F_k(x, u) &= N^{\frac{1}{2}} u \text{diag}[\overline{\Phi_k^1(x)}, \overline{\Phi_k^2(x)}, \dots, \overline{\Phi_k^N(x)}] u^* \\ &= N^{\frac{1}{2}} \sum_{i=1}^N u \overline{\Phi_k^i(x)} \mathbb{P}_i u^*, \quad (x, u) \in X \times SU(N), \end{aligned} \quad (3.8)$$

where the \mathbb{P}_i are the one-dimensional projection operators, $\chi^i \chi^{i\dagger}$, built out of the vectors χ^i in the chosen orthonormal basis of \mathbb{C}^N . It is then not hard to see, using the orthonormality of the vectors $\{\Phi_i\}$ and the relation (3.7), that

$$\int_{X \times SU(N)} F_k(x) F_\ell(x)^* d\mu(x) d\Omega(u) = \mathbb{I}_N \delta_{k\ell}, \quad k, \ell = 1, 2, \dots, N. \quad (3.9)$$

Referring to the general construction of MVCS in Section II.1, we take $\mathcal{A} = \mathcal{B} = \mathcal{M}_N(\mathbb{C})$ and $\mathbf{E} = \mathcal{M}_N(\mathbb{C})$, considered as a Hilbert module over itself. We take \mathbf{G} to be the Hilbert space $\mathfrak{H}_{\mathbf{K}}$, considered as a Hilbert module over \mathbb{C} . The MVCS are then defined as:

$$|x, u, V\rangle = \sum_k V F_k(x, u) \otimes \Phi_k \in \mathbf{H} = \mathcal{M}_N(\mathbb{C}) \otimes \mathfrak{H}_{\mathbf{K}}, \quad \text{for all } (x, u) \in X \times SU(N), \quad (3.10)$$

where V is a unitary element in $SU(N)$. These MVCS satisfy the resolution of the identity,

$$\int_{X \times SU(N)} |x, u, V\rangle \langle x, u, V| d\mu(x) d\Omega(u) = I_{\mathbf{H}}.$$

In order to recover the VCS (3.6) from here, we use the projection operators, $\mathbb{P}_i(u) = u \mathbb{P}_i u^*$, $u \in SU(N)$ and simply take the partial trace in $\mathcal{B} = \mathcal{M}_N(\mathbb{C})$,

$$|x, i\rangle = \text{Tr}_{\mathcal{B}}[\mathbb{P}_i(u) |x, u, \mathbb{I}_N\rangle]. \quad (3.11)$$

A related example is that of the analytic VCS, built in [6], using powers of matrices from $\mathcal{M}_N(\mathbb{C})$. These VCS may be defined as:

$$|\mathfrak{z}, i\rangle = \sum_k \frac{\mathfrak{z}^k}{\sqrt{c_k}} \chi^i \otimes \Phi_k \quad \mathfrak{z} \in \mathcal{M}_N(\mathbb{C}), \quad (3.12)$$

where the c_k are the numbers (see, e.g. [6, 12]),

$$c_k = \frac{1}{(k+1)(k+2)} \left[\prod_{j=1}^{k+1} (N+j) - \prod_{j=1}^{k+1} (N-j) \right], \quad k = 0, 1, 2, \dots,$$

Let z_{ij} , $i, j = 1, 2, \dots, N$ be the matrix elements of \mathfrak{Z} . Then, writing

$$F_k(\mathfrak{Z}) = \frac{\mathfrak{Z}^k}{\sqrt{c_k}} \quad \text{and} \quad z_{ij} = x_{ij} + iy_{ij},$$

it can be shown that,

$$\int_{\mathcal{M}_N(\mathbb{C})} F_k(\mathfrak{Z}) F_\ell(\mathfrak{Z})^* d\mu(\mathfrak{Z}, \mathfrak{Z}^*) = \delta_{k\ell} \mathbb{I}_N, \quad d\mu(\mathfrak{Z}, \mathfrak{Z}^*) = \frac{e^{-\text{Tr}[\mathfrak{Z}^* \mathfrak{Z}]}}{(2\pi)^N} \prod_{i,j=1}^N dx_{ij} dy_{ij}.$$

Using this fact, one may prove the resolution of the identity,

$$\sum_{i=1}^N \int_{\mathcal{M}_N(\mathbb{C})} |\mathfrak{Z}, i\rangle \langle \mathfrak{Z}, i| d\mu(\mathfrak{Z}, \mathfrak{Z}^*) = \mathbb{I}_N \otimes I_{\mathfrak{H}_{\mathbf{K}}}.$$

To construct the related MVCS, we consider $\mathcal{M}_N(\mathbb{C})$ as a module over itself and identify it with \mathbf{E} . The module \mathfrak{H} , containing the functions F_k , then consists of functions from $\mathcal{M}_N(\mathbb{C})$ to itself. Considering $\mathfrak{H}_{\mathbf{K}}$ as a module over \mathbb{C} , we may define MVCS in $\mathbf{H} = \mathcal{M}_N(\mathbb{C}) \otimes \mathfrak{H}_{\mathbf{K}}$ as

$$|\mathfrak{Z}, a\rangle = \sum_k a F_k(\mathfrak{Z}) \otimes \Phi_k = \sum_k a \frac{\mathfrak{Z}^k}{\sqrt{c_k}} \otimes \Phi_k, \quad (3.13)$$

where a is a unitary element in $\mathcal{M}_N(\mathbb{C})$. These then satisfy the resolution of the identity,

$$\int_{\mathcal{M}_N(\mathbb{C})} |\mathfrak{Z}, a\rangle \langle \mathfrak{Z}, a| d\mu(\mathfrak{Z}, \mathfrak{Z}^*) = I_{\mathbf{H}}. \quad (3.14)$$

In the particular case when $N = 2$ the set $\mathcal{M}_N(\mathbb{C})$, of all complex 2×2 matrices, can be identified with the space of *complex quaternions*. The resulting MVCS may then be called *complex quaternionic MVCS*.

Example 3. A real quaternionic variant

Vector coherent states of the type (3.12), when \mathfrak{Z} is replaced by a *real* quaternionic variable \mathfrak{q} , have been constructed in [5] and [15], while coherent states in quaternionic Hilbert spaces have been studied in [1]. These latter coherent states, which are a natural generalization of the canonical coherent states to quaternionic quantum mechanics [2], have also been shown to have interesting physical applications. We now construct an analogous family of quaternionic coherent states on a quaternionic Hilbert space. Recall that a quaternionic Hilbert space is a linear vector space over the field of (real) quaternions, \mathbb{H} , with the inner product taking

values in \mathbb{H} . While \mathbb{H} contains the complexes, it is not a C^* -algebra. So strictly speaking, a quaternionic Hilbert space is not a Hilbert C^* -module. However, the quaternionic CS we shall now construct are very similar to the MVCS (3.13).

Let us start with the quaternionic vector coherent states introduced in [15]. These are vector coherent states defined on a standard Hilbert space \mathfrak{H} (over the complexes). We take for $\mathbf{q} \in \mathbb{H}$ its representation by 2×2 complex marices:

$$\mathbf{q} = u(\theta, \phi) \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} u(\theta, \phi)^* , \quad u(\theta, \phi) = \begin{pmatrix} ie^{i\frac{\phi}{2}} \cos \frac{\theta}{2} & -e^{i\frac{\phi}{2}} \sin \frac{\theta}{2} \\ e^{-i\frac{\phi}{2}} \sin \frac{\theta}{2} & -ie^{-i\frac{\phi}{2}} \cos \frac{\theta}{2} \end{pmatrix} \quad (3.15)$$

where $z \in \mathbb{C}$, $0 \leq \theta \leq \pi$, $0 < \phi \leq 2\pi$. Writing $z = re^{i\xi}$, we also have,

$$\mathbf{q} = r[\mathbb{I}_2 \cos \xi + i\sigma(\hat{n}) \sin \xi] = re^{i\xi\sigma(\hat{n})}, \quad (3.16)$$

with

$$\mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma(\hat{n}) = \begin{pmatrix} \cos \theta & e^{i\phi} \sin \theta \\ e^{-i\phi} \sin \theta & -\cos \theta \end{pmatrix}, \quad [\sigma(\hat{n})]^2 = \mathbb{I}_2$$

Let $\{\Psi_n\}_{n=0}^\infty$ be an orthonormal basis of \mathfrak{H} and χ^i , $i = 1, 2$, an orthonormal basis of \mathbb{C}^2 . Normalized *quaternionic vector coherent states* are then defined [5, 15] as

$$|\mathbf{q}, j\rangle = \frac{e^{-\frac{r^2}{2}}}{\sqrt{2}} \sum_{n=0}^\infty \frac{\mathbf{q}^n}{\sqrt{n!}} \chi^i \otimes \Psi_n \in \mathbb{C}^2 \otimes \mathfrak{H}, \quad \sum_{j=1}^2 \| |\mathbf{q}, j\rangle \|^2 = 1. \quad (3.17)$$

These vectors satisfy the resolution of the identity,

$$\sum_{j=1}^2 \int_{\mathbb{H}} |\mathbf{q}, j\rangle \langle \mathbf{q}, j| d\mu(\mathbf{q}, \mathbf{q}^\dagger) = \mathbb{I}_2 \otimes I_{\mathfrak{H}}, \quad d\mu(\mathbf{q}, \mathbf{q}^\dagger) = \frac{1}{8\pi^2} r dr d\xi \sin \theta d\theta d\phi. \quad (3.18)$$

Suppose now that $\mathfrak{H}_{\text{quat}}$ is a Hilbert space over the quaternions. (Multiplication by elements of \mathbb{H} from the right is assumed, i.e., if $\Phi \in \mathfrak{H}_{\text{quat}}$ and $\mathbf{q} \in \mathbb{H}$, then $\Phi\mathbf{q} \in \mathfrak{H}_{\text{quat}}$). The obvious generalization of the VCS (3.17) to *quaternionic coherent states* over $\mathfrak{H}_{\text{quat}}$ are easily written down by taking an orthonormal basis $\{\Psi_n^{\text{quat}}\}_{n=0}^\infty$ in $\mathfrak{H}_{\text{quat}}$ and defining the vectors

$$|\mathbf{q}\rangle = e^{-\frac{r^2}{2}} \sum_{n=0}^\infty \Psi_n^{\text{quat}} \frac{\mathbf{q}^n}{\sqrt{n!}} \in \mathfrak{H}_{\text{quat}}, \quad \mathbf{q} \in \mathbb{H}, \quad \langle \mathbf{q} | \mathbf{q} \rangle_{\mathfrak{H}_{\text{quat}}} = \mathbb{I}_2. \quad (3.19)$$

They satisfy the resolution of the identity,

$$\int_{\mathbb{H}} |\mathbf{q}\rangle \langle \mathbf{q}| d\nu(\mathbf{q}, \mathbf{q}^\dagger) = I_{\mathfrak{H}_{\text{quat}}}, \quad d\nu(\mathbf{q}, \mathbf{q}^\dagger) = \frac{1}{4\pi^2} r dr d\xi \sin \theta d\theta d\phi. \quad (3.20)$$

These coherent states were obtained in [1], where a group theoretical argument was used to construct them. Recently they have also been obtained in [16]. Here we stress their similarity with our general construction over C^* -modules.

Example 4. Infinite component VCS

As a similar example to the above, but this time involving VCS with an infinite number of components, we consider the VCS

$$|z, \bar{z}'; \ell\rangle = e^{-\frac{1}{2}(|z'|^2 + |z|^2)} \bar{z}'^\ell \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n! \ell!}} |\Psi_n\rangle, \quad \ell = 0, 1, 2, \dots, \infty, \quad (z, \bar{z}') \in \mathbb{C} \times \mathbb{C}. \quad (3.21)$$

where the Ψ_n form an orthonormal basis in some Hilbert space \mathfrak{H} . These VCS are similar to those appearing in the problem of an electron moving in a constant magnetic field and its associated *Landau levels* [4]. They satisfy the normalization condition,

$$\sum_{\ell=0}^{\infty} \langle z, \bar{z}'; \ell | z, \bar{z}'; \ell \rangle = 1,$$

and the resolution of the identity,

$$\sum_{\ell=0}^{\infty} \int_{\mathbb{C}} |z, \bar{z}'; \ell\rangle \langle z, \bar{z}'; \ell| \frac{dx dy}{\pi} = I_{\mathfrak{H}}, \quad z = x + iy. \quad (3.22)$$

In order to construct a family of MVCS corresponding to this set of VCS, we start with a locally compact, unimodular group G (such as, e.g., $SU(1, 1)$), which has a representation, in the discrete series, in an *infinite dimensional* Hilbert space \mathfrak{K} . Let $G \ni g \mapsto U(g)$ be such a unitary irreducible representation and let $d\mu_G$ denote the Haar measure of G . It is then well-known (see, e.g., [3]) that if ϕ is any unit vector in \mathfrak{K} , then

$$\frac{1}{d} \int_G U(g) |\phi\rangle \langle \phi| U(g)^* d\mu_G(g) = I_{\mathfrak{K}}, \quad (3.23)$$

where $d > 0$ is a constant, called the *formal dimension* of the representation U . Let $\{\phi_i\}_{i=1}^{\infty}$ be an orthonormal basis of \mathfrak{K} and $\mathbb{P}_i = |\phi_i\rangle \langle \phi_i|$ the corresponding one-dimensional projection operators. We define the functions, $F_k : \mathbb{C} \times \mathbb{C} \times G \longrightarrow \mathcal{L}(\mathfrak{K})$:

$$F_k(z, \bar{z}', g) = \frac{1}{d^{\frac{1}{2}}} e^{-(|z'|^2 + |z|^2)} \frac{z^k}{\sqrt{k!}} \sum_{n=1}^{\infty} \frac{\bar{z}'^n}{\sqrt{n!}} \mathbb{P}_n(g), \quad \mathbb{P}_n(g) = U(g) \mathbb{P}_n U(g)^*. \quad (3.24)$$

It is then easy to see that,

$$\int_{\mathbb{C} \times G} F_k(z, \bar{z}', g) F_\ell(z, \bar{z}', g)^* \frac{dx dy}{\pi} d\mu_G(g) = \delta_{k\ell} I_{\mathfrak{K}}, \quad z = x + iy.$$

Thus, considering $\mathcal{L}(\mathfrak{K})$ as a C^* -algebra and as a Hilbert module over itself, we again define the MVCS on $\mathbf{H} = \mathcal{L}(\mathfrak{K}) \otimes \mathfrak{H}$,

$$|z, \bar{z}', g; a\rangle = \sum_{k=1}^{\infty} a F_k(z, \bar{z}', g) \otimes \Psi_k = \frac{1}{d^{\frac{1}{2}}} e^{-(|z'|^2 + |z|^2)} \sum_{k,n} \frac{\bar{z}'^n z^k}{\sqrt{n! k!}} a \mathbb{P}_n(g) \otimes \Psi_k,$$

where, once more, a is a unitary element in $\mathcal{L}(\mathfrak{K})$ and $\{\Psi_k\}_{k=1}^\infty$ an orthonormal basis of \mathfrak{H} . These MVCS clearly have all the required properties, e.g., the resolution of the identity,

$$\int_{\mathbb{C} \times G} |z, \bar{z}', g; a\rangle \langle z, \bar{z}', g; a| \frac{dx dy}{\pi} d\mu_G(g) = I_{\mathfrak{H}},$$

and the VCS can be obtained from them by taking the partial trace in $\mathcal{L}(\mathfrak{K})$:

$$|z, \bar{z}', \ell\rangle = \text{Tr}_{\mathcal{L}(\mathfrak{K})} [\mathbb{P}_\ell(g) |z, \bar{z}', g; I_{\mathfrak{K}}\rangle].$$

In the next example we construct a different variant of these MVCS, using Cuntz algebras.

Example 5. Coherent states from representations of Cuntz algebras

Let S_1, S_2, \dots be isometries on a complex separable Hilbert space \mathcal{K} (necessarily infinite dimensional) such that

$$\sum_{j=1}^{\infty} S_j S_j^* = I_{\mathcal{K}}$$

where the sum converges in the strong operator topology of $\mathcal{B}(\mathcal{K})$. Multiplying both sides by S_i^* , we get

$$S_i^* + S_i^* \sum_{j \neq i} S_j S_j^* = S_i^*$$

so that

$$S_i^* \sum_{j \neq i} S_j S_j^* = 0$$

But $\sum_{j \neq i} S_j S_j^*$ is the projection onto the closure of the span of the ranges of S_j for $j \neq i$. So the range of S_i is orthogonal to the range of S_j for all $j \neq i$. This is a representation of the Cuntz algebra \mathcal{O}_∞ with infinitely many generators. Take $\mathbf{G} = \mathcal{C}$ to be the C^* -algebra generated by the isometries S_1, S_2, \dots . Choose $\phi_i = S_i$. Then

$$\langle \phi_i, \phi_j \rangle = S_i^* S_j = \delta_{ij} I_{\mathcal{K}} \text{ and } \sum_{k=1}^{\infty} |\phi_k\rangle \langle \phi_k| = I.$$

Our coherent states are

$$|x, a\rangle = \left(\sum_{k=1}^{\infty} a \cdot F_k(x) \otimes S_k \right) (\mathcal{N}(x)^{-1/2} \otimes I).$$

We now construct an explicit example of a Cuntz algebra. Let $\omega : \mathbb{N}^{>0} \longrightarrow \mathbb{N}^{>0} \times \mathbb{N}^{>0}$ be a bijection ($\mathbb{N}^{>0}$ denoting the set of non-zero, positive integers). Consider a Hilbert space \mathfrak{H} and let $\{\phi_n\}_{n \in \mathbb{N}^{>0}}$ be an orthonormal basis of it. Writing $\omega(n) = (k, \ell)$ we define a re-transcription of this basis in the manner

$$\psi_{k\ell} := \phi_n = \psi_{\omega(n)}, \quad k, n, \ell \in \mathbb{N}^{>0}. \quad (3.25)$$

Note that the two sets of vectors are exactly the same and satisfy, $\langle \phi_m | \phi_n \rangle_{\mathfrak{H}} = \delta_{mn}$ and $\langle \psi_{mn} | \psi_{k\ell} \rangle_{\mathfrak{H}} = \delta_{mk} \delta_{n\ell}$, respectively. Define the family of isometries S_k , $k \in \mathbb{N}^{>0}$ on \mathfrak{H} , in the manner

$$S_k \phi_n = \psi_{kn} , \quad n \in \mathbb{N}^{>0} . \quad (3.26)$$

Note that this defines an isometry and not a unitary map. Indeed, one has,

$$S_k^* S_\ell = \delta_{k\ell} I_{\mathfrak{H}} \quad \text{and} \quad \sum_{k \in \mathbb{N}^{>0}} S_k S_k^* = \sum_{k \in \mathbb{N}^{>0}} \mathbb{P}_k = I_{\mathfrak{H}} , \quad (3.27)$$

\mathbb{P}_k being the projection operator onto the subspace \mathfrak{H}_k of \mathfrak{H} spanned by the vectors $\psi_{k\ell}$, $\ell \in \mathbb{N}^{>0}$. Moreover, $S_k S_\ell^*$ is a partial isometry from \mathfrak{H}_ℓ to \mathfrak{H}_k .

The C^* -algebra \mathcal{O}_∞ , generated by these isometries, is then a Cuntz algebra. An explicit example of such a bijection ω is given in Appendix V.2.

The above construction has an immediate application to a physical situation. We consider the non-normalized version (with a set to the unit element of \mathcal{A}),

$$|x\rangle = \sum_{k=1}^{\infty} F_k(x) \otimes S_k .$$

Let $X = \mathbb{C}$ and $\mathbf{E} = L^2(\mathbb{C}, \frac{e^{-|z|^2}}{2\pi} dx dy)$, $z = \frac{1}{\sqrt{2}}(x + iy)$. We take $F_k : \mathbb{C} \rightarrow \mathbb{C}$ to be the functions,

$$F_k(z) = \frac{z^{k-1}}{\sqrt{(k-1)!}} , \quad k = 1, 2, 3, \dots$$

Let $\psi_{k\ell}$ be the complex Hermite polynomials,

$$\psi_{k\ell}(\bar{z}, z) = \frac{(-1)^{n+k-2}}{\sqrt{(\ell-1)!(k-1)!}} e^{|z|^2} \partial_{\bar{z}}^{\ell-1} \partial_z^{k-1} e^{-|z|^2} , \quad k, \ell = 1, 2, 3, \dots \quad (3.28)$$

These form an orthonormal basis of $L^2(\mathbb{C}, \frac{e^{-|z|^2}}{2\pi} dx dy)$. The coherent states now become

$$|z\rangle = \sum_{k=1}^{\infty} \frac{z^{k-1}}{\sqrt{(k-1)!}} S_k . \quad (3.29)$$

Let ϕ_n be as in (3.25) and consider the vectors

$$\xi_{\bar{z}', n} = \frac{\bar{z}'^{n-1}}{\sqrt{(n-1)!}} \phi_n .$$

Then the vectors (in $L^2(\mathbb{C}, \frac{e^{-|z|^2}}{2\pi} dx dy)$),

$$|z, \bar{z}', n\rangle = \sum_{k=1}^{\infty} \frac{z^{k-1}}{\sqrt{(k-1)!}} S_k \xi_{\bar{z}', n} = \bar{z}'^{n-1} \sum_{k=1}^{\infty} \frac{z^{k-1}}{\sqrt{(k-1)! (n-1)!}} \psi_{kn} , \quad (3.30)$$

($\ell = 1, 2, 3, \dots, \infty$) are just the non-normalized versions of the infinite component vector CS associated to the Landau levels, found in [4].

IV Reproducing kernel, carrier space and a minimal dilation

We start with a brief description of a *completely positive kernel*. Given two C^* -algebras \mathcal{A} and \mathcal{B} , an $(\mathcal{A}, \mathcal{B})$ -*reproducing kernel correspondence* on a set X , is an $(\mathcal{A}, \mathcal{B})$ -correspondence E whose elements are \mathcal{B} -valued functions $f: (x, a) \mapsto f(x, a) \in \mathcal{B}$ on $X \times \mathcal{A}$. It is a vector space with respect to the usual pointwise vector space operations. Moreover, there is a *kernel element* $k_x \in E$ such that

$$f(x, a) = \langle k_x, a \cdot f \rangle_E \quad (4.1)$$

for every $x \in X$. When this is the case we say that the function $K: X \times X \rightarrow \mathcal{L}(\mathcal{A}, \mathcal{B})$ given by

$$K(x, y)[a] = k_x(y, a) \quad (4.2)$$

is the *reproducing kernel* for the reproducing kernel correspondence E .

From the inner product characterization in (4.1) of the point evaluation for elements in an $(\mathcal{A}, \mathcal{B})$ -reproducing kernel correspondence E on X one easily deduces that the left \mathcal{A} -action and the right \mathcal{B} -action are given by

$$(a \cdot f)(x', a') = f(x', a'a) \text{ and } (f \cdot b)(x', a') = f(x', a')b. \quad (4.3)$$

The mapping from \mathcal{A} to \mathcal{B} given by $a \mapsto f(x, a)$ is \mathcal{A} -linear for each fixed $f \in E$ and $x \in X$. Since all our algebras are unital, this follows from the general identity $f(x, a) = (a \cdot f)(x, 1_{\mathcal{A}})$ (a consequence of (4.1), (4.3) and the linearity of the point-evaluation map $f \mapsto f(x, 1_{\mathcal{A}})$). Note also that we recover the element k_x from K by using formula (4.2) to define k_x as a function of (y, a) for each $x \in X$.

Given a reproducing kernel $(\mathcal{A}, \mathcal{B})$ -correspondence, one can show that the associated reproducing kernel function $K: X \times X \rightarrow \mathcal{L}(\mathcal{A}, \mathcal{B})$ defined by (4.2) is a completely positive kernel in the sense of [8], i.e., the function

$$((x, a), (x', a')) \rightarrow K(x, x')[a^* a']$$

is a positive C^* -algebra valued kernel. This means that

$$\sum_{i,j=1}^N b_i^* K(x_i, x_j)[a_i^* a_j] b_j$$

is a positive element of \mathcal{B} for each choice of finitely many $(x_1, a_1), \dots, (x_N, a_N)$ in $X \times \mathcal{A}$ and b_1, \dots, b_N in \mathcal{B} . The following theorem, which again can be found in [7] gives a complete clarity.

Theorem IV.1 *Given a function $K: X \times X \rightarrow \mathcal{L}(\mathcal{A}, \mathcal{B})$, the following are equivalent:*

1. *K is a completely positive kernel in the sense that the function from $(X \times \mathcal{A}) \times (X \times \mathcal{A}) \rightarrow \mathcal{L}(\mathcal{A}, \mathcal{B})$ given by*

$$((x, a), (x', a')) \mapsto K(x', x)[a^* a']$$

is a positive kernel in the sense that

$$\sum_{i,j=1}^N b_i^* K(x_i, x_j)[a_i^* a_j] b_j \geq 0 \text{ in } \mathcal{B}$$

for all $(x_1, a_1), \dots, (x_N, a_N) \in X \times \mathcal{A}$ and $b_1, \dots, b_N \in \mathcal{B}$.

2. *K has a Kolmogorov decomposition in the sense of [8], i.e., there exists an $(\mathcal{A}, \mathcal{B})$ -correspondence E and a mapping $x \mapsto k_x$ from X into E such that*

$$K(x, y)[a] = \langle k_x, a \cdot k_y \rangle_E \text{ for all } a \in \mathcal{A}.$$

3. *K is the reproducing kernel for an $(\mathcal{A}, \mathcal{B})$ -reproducing kernel correspondence $E = E(K)$, i.e., there is an $(\mathcal{A}, \mathcal{B})$ -correspondence $E = E(K)$ whose elements are \mathcal{B} -valued functions on $X \times \mathcal{A}$ such that the function $k_x: (x', a') \mapsto K(x', x)[a']$ is in $E(K)$ for each $x \in X$ and has the reproducing property*

$$\langle k_x, a \cdot f \rangle_{E(K)} = \langle a^* \cdot k_x, f \rangle_{E(K)} = f(x, a) \text{ for all } x \in X \text{ and } a \in \mathcal{A}$$

where $a^ \cdot k_x$ is given by*

$$(a^* \cdot k_x)(x', a') = K(x', x)[a^* a'] = \langle a^* \cdot k_{x'}, a' \cdot k_x \rangle. \quad (4.4)$$

Corresponding to the MVCS in (2.15), define the kernel $K: X \times X \rightarrow \mathcal{L}(\mathcal{A}, \mathcal{B} \otimes \mathcal{C})$ by

$$K(x, y)a^* a' = \langle x, a \mid y, a' \rangle_{\mathbf{H}} = \sum_k \langle a \cdot F_k(x) \otimes \phi_k \mid a' \cdot F_k(y) \otimes \phi_k \rangle_{\mathbf{H}}, \quad (4.5)$$

for all $x, y \in X$ and $a, a' \in \mathcal{A}$. This is a completely positive kernel. In fact, (4.5) gives the Kolmogorov decomposition of the kernel.

Let us see what reproducing property this kernel has.

$$\begin{aligned} k(x, z)a^* a' &= \langle x, a \mid z, a' \rangle \\ &= \langle x, a \mid I_{\mathbf{H}} \mid z, a' \rangle \\ &= \langle x, a \mid \int_X \mid y, b \rangle \langle y, b \mid d\mu(y) \mid z, a' \rangle \text{ for a co-isometry } b \in \mathcal{A} \\ &= \int_X \langle x, a \mid y, b \rangle \langle y, b \mid z, a' \rangle d\mu(y) \\ &= \int_X k(x, y)a^* b \ k(y, z)b^* a' d\mu(y). \end{aligned}$$

In particular, taking $a = b = \text{id}_{\mathcal{A}}$, we get that $k(x, z)a' = \int_X k(x, y)\text{id}_{\mathcal{A}} k(y, z) d\mu(y) a'$ which means that $k(x, z) = \int_X k(x, y)\text{id}_{\mathcal{A}} k(y, z) d\mu(y)$.

We show the existence of an associated reproducing kernel and a carrier Hilbert module, corresponding to a family of MVCS. Going back to the setting of Section II, we take again the measure space (X, μ) , the Hilbert modules \mathbf{E} over the C^* -algebra \mathcal{B} , \mathbf{G} over the C^* -algebra \mathcal{C} , $\mathbf{H} = \mathbf{E} \otimes \mathbf{G}$ and the Hilbert module \mathfrak{H} , consisting of measurable functions $F : X \mapsto \mathbf{E}$, which satisfy the “square integrability condition”,

$$\left\| \int_X \langle F(x) \mid F(x) \rangle_{\mathbf{E}} d\mu(x) \right\|_{\mathcal{B}} < \infty .$$

We also require that the elements $\phi_i \in \mathbf{G}$ used to define the non-normalized MVCS in (2.15) satisfy both the conditions (2.14) and (2.18).

Consider now the Hilbert module $\tilde{\mathfrak{H}}$, over $\mathcal{B} \otimes \mathcal{C}$, consisting of maps $\tilde{\mathbf{h}} : X \mapsto \mathcal{B} \otimes \mathcal{C}$, under the (module) inner product,

$$\langle \tilde{\mathbf{h}}_1 \mid \tilde{\mathbf{h}}_2 \rangle_{\tilde{\mathfrak{H}}} = \int_X \tilde{\mathbf{h}}_1(x)^* \tilde{\mathbf{h}}_2(x) d\mu(x) .$$

Recall that our coherent states $|x, a\rangle$ in (2.15) are elements of the Hilbert module \mathbf{H} . Using these MVCS we now define the linear map $W : \mathbf{H} \mapsto \tilde{\mathfrak{H}}$ by

$$(Wf)(x) = \langle x, \text{id}_{\mathcal{A}} \mid f \rangle_{\mathbf{H}} , \quad f \in \mathbf{H} . \quad (4.6)$$

That W is an isometry is then clear, since

$$\langle Wf \mid Wf \rangle_{\tilde{\mathfrak{H}}} = \int_X \langle f \mid x, \text{id}_{\mathcal{A}} \rangle \langle x, \text{id}_{\mathcal{A}} \mid f \rangle d\mu(x) = \langle f \mid f \rangle_{\mathbf{H}} ,$$

using (2.16).

Theorem IV.2 *The range of W is a complemented submodule of $\tilde{\mathfrak{H}}$.*

Proof. We denote by \mathbb{P}_K the linear operator on $\tilde{\mathfrak{H}}$ defined by

$$(\mathbb{P}_K \tilde{\mathbf{h}})(x) = \int_X K(x, y) \tilde{\mathbf{h}}(y) d\mu(y) , \quad \text{for all } \tilde{\mathbf{h}} \in \tilde{\mathfrak{H}} . \quad (4.7)$$

It is then straightforward to verify that \mathbb{P}_K is a projection in the C^* -algebra $\mathcal{L}(\tilde{\mathfrak{H}})$ and the range of the isometry in $\tilde{\mathfrak{H}}$, which we denote by \mathfrak{H}_K , is range of the projection \mathbb{P}_K . The range of a projection is always a complemented submodule. \blacksquare

We call it a *reproducing kernel submodule* because it is the image under an isometry of an \mathcal{A} - \mathcal{B} correspondence. The reproducing kernel $K(x, y)$ is $\langle x, \text{id}_{\mathcal{A}} \mid y, \text{id}_{\mathcal{A}} \rangle$. It follows that

$$\tilde{\mathfrak{H}} = \mathfrak{H}_K \oplus \mathfrak{H}_K^\perp .$$

Writing

$$\mathbf{h}_x = W |x, \text{id}_A\rangle \in \tilde{\mathfrak{H}}, \quad \text{so that} \quad K(x, y) = \langle \mathbf{h}_x | \mathbf{h}_y \rangle_{\tilde{\mathfrak{H}}} = \mathbf{h}_y(x) \in \mathcal{B} \otimes \mathcal{C}, \quad (4.8)$$

we see that the vectors \mathbf{h}_x span the submodule \mathfrak{H}_K . From (2.16), (4.7) and (4.8), it also follows that

$$\int_X |\mathbf{h}_x\rangle \langle \mathbf{h}_x| d\mu(x) = \mathbb{P}_K. \quad (4.9)$$

Note that the vectors \mathbf{h}_x , $x \in X$, being unitary images in \mathfrak{H}_K of the MVCS $|x, \text{id}_A\rangle$, are also themselves MVCS. Furthermore, the submodule \mathfrak{H}_K has a natural left action for $a \in \mathcal{A}$ given by

$$(a \cdot \tilde{\mathbf{h}})(x) = (Waf)(x), \quad \text{where} \quad \tilde{\mathbf{h}} = Wf.$$

Finally, using the \mathbf{h}_x we may define a POV measure on \mathfrak{H}_K and obtain a natural dilation of it to a PV measure on $\tilde{\mathfrak{H}}$. Indeed, the POV measure is defined on the Borel sets Δ of X as,

$$\nu(\Delta) = \int_{\Delta} |\mathbf{h}_x\rangle \langle \mathbf{h}_x| d\mu(x) \in \mathcal{L}(\mathfrak{H}_K), \quad (4.10)$$

and the PV measure $\tilde{P}(\Delta)$ by

$$(\tilde{P}(\Delta)\tilde{\mathbf{h}})(x) = \chi_{\Delta}(x)\tilde{\mathbf{h}}(x), \quad \tilde{\mathbf{h}} \in \tilde{\mathfrak{H}}, \quad (4.11)$$

χ_{Δ} being the characteristic function of the set Δ . It is then straightforward to verify that

$$\nu(\Delta) = \mathbb{P}_K \tilde{P}(\Delta) \mathbb{P}_K. \quad (4.12)$$

If X is a locally compact space and the support of the measure μ is assumed to be the whole of X (i.e., no open set has measure zero), this dilation can easily be shown to be minimal, in the sense of Naimark. In other words, the set of vectors of the type $\tilde{P}(\Delta)\tilde{\mathbf{h}}$, as Δ runs through all Borel sets and $\tilde{\mathbf{h}}$ through $\tilde{\mathfrak{H}}$, spans $\tilde{\mathfrak{H}}$. The proof is an easy adaptation of the proof of the analogous result for Hilbert spaces (see, e.g., [3, p. 36]).

The space $\tilde{\mathfrak{H}}$ acts as a carrier space for the MVCS, the situation with the dilation here being exactly the same as on a Hilbert space. It would appear that the fundamental ingredients for the existence of a family of MVCS are (i) a Hilbert C^* -module of the type $\tilde{\mathfrak{H}}$, consisting of functions from a finite measure space (X, μ) to the C^* -algebra defining the module and (ii) a reproducing kernel submodule contained in this Hilbert module. We plan to discuss these issues in greater detail in a succeeding publication.

V Appendix

V.1 Hilbert C^* -modules

In this appendix we collect together some preliminary notions and results on Hilbert modules. Let A be a C^* -algebra (not necessarily unital or commutative). An *inner-product A -module* is a linear space E which is a right A -module (with compatible scalar multiplication: $\lambda(xa) = (\lambda x)a = x(\lambda a)$ for $x \in E, a \in A, \lambda \in \mathbb{C}$), together with a map $(x, y) \mapsto \langle x \mid y \rangle$ from $E \times E \rightarrow A$ such that

- (i) $\langle x \mid \alpha y + \beta z \rangle = \alpha \langle x \mid y \rangle + \beta \langle x \mid z \rangle$ $(x, y, z \in E, \alpha, \beta \in \mathbb{C})$,
- (ii) $\langle x \mid ya \rangle = \langle x \mid y \rangle a$ $(x, y \in E, a \in A)$,
- (iii) $\langle y \mid x \rangle = \langle x \mid y \rangle^*$ $(x, y \in E)$,
- (iv) $\langle x \mid x \rangle \geq 0$; if $\langle x \mid x \rangle = 0$ then $x = 0$.

Note that in condition (i) the inner-product is required to be linear in its *second* variable. From (iii) it follows that the inner-product is conjugate-linear in its first variable. We adopt the same convention for ordinary inner-product spaces and Hilbert spaces (so that an inner-product space is the same thing as an inner-product \mathbb{C} -module). If E satisfies all the conditions for an inner-product A -module except for the second part of condition (iv) then we call E a *semi-inner-product A -module*. For such modules we have a version of Cauchy-Schwarz inequality:

Proposition 1 [13, p. 3] *If E is a semi-inner-product A -module and $x, y \in E$ then*

$$\langle y \mid x \rangle \langle x \mid y \rangle \leq \|\langle x \mid x \rangle\| \|\langle y \mid y \rangle\|$$

For $x \in E$ we write $\|x\| = \|\langle x \mid x \rangle\|^{\frac{1}{2}}$. It follows from Proposition 1 that $\|\langle x \mid y \rangle\| \leq \|x\| \|y\|$ and it is easy to deduce from this that if E is an inner-product A -module then $\|\cdot\|$ is a norm on E . An inner-product A -module which is complete with respect to its norm is called a *Hilbert A -module*, or a *Hilbert C^* -module over the C^* -algebra A* .

One would like to think that Hilbert C^* -modules behave like Hilbert spaces, and in some ways they do. For example, if E is a Hilbert A -module and $x \in E$ then it is easy to check that

$$\|x\| = \sup \{ \|\langle x \mid y \rangle\| : y \in E, \|y\| \leq 1 \}.$$

But there is a fundamental way in which Hilbert C^* -modules differ from Hilbert spaces. Given a closed submodule F of a Hilbert A -module E , define

$$F^\perp = \{ y \in E : \langle x \mid y \rangle = 0, (x \in F) \}$$

Then F^\perp is also a closed submodule of E . But E is not (usually) equal to $F \oplus F^\perp$ (and $F^{\perp\perp}$ is usually bigger than F) [13, p. 7].

Suppose that E, F are Hilbert A -modules. We define $\mathcal{L}(E, F)$ to be the set of all linear (\mathbb{C} -linear) maps $t : E \longrightarrow F$ for which there is a linear map $t^* : F \longrightarrow E$ such that

$$\langle tx \mid y \rangle = \langle x \mid t^*y \rangle \quad (x \in E, y \in F).$$

It is easy to see that t must be A -linear (that is, t is linear and $t(xa) = t(x)a$ for all $x \in E, a \in A$) and bounded as a map between the Banach spaces E and F . We call $\mathcal{L}(E, F)$ the set of *adjointable* maps from E and F . In particular, $\mathcal{L}(E, E)$ which we abbreviate to $\mathcal{L}(E)$, is a C^* -algebra. Thus every element of $\mathcal{L}(E, F)$ is a bounded A -linear map. But the converse is false: a bounded A -linear map need not be adjointable [13, p. 8].

We say that a closed submodule F of a Hilbert A -module is *complemented* if $F \oplus F^\perp$. As already emphasized that a closed submodule of a Hilbert C^* -module need not be complemented. The following theorem enables us to conclude that certain submodules are complemented.

Theorem 2 [13, p. 22] *Let E, F be Hilbert A -modules and suppose that t in $\mathcal{L}(E, F)$ has closed range. Then*

- (i) $\ker(t)$ is a complemented submodule of E .
- (ii) $\text{ran}(t)$ is a complemented submodule of F .
- (iii) the mapping $t^* \in \mathcal{L}(E, F)$ also has closed range.

An operator $u \in \mathcal{L}(E, F)$ is said to be a *unitary* if $u^*u = 1_E$ and $uu^* = 1_F$. If there exists a unitary element of $\mathcal{L}(E, F)$ then we say that E and F are *unitarily equivalent* Hilbert A -modules, and we write $E \approx F$. The following two results characterise unitary maps and isometries from Hilbert A -modules E to F respectively.

Theorem 3 [13, p. 26] *Let A be a C^* -algebra, let E, F be Hilbert A -modules and u be a linear map from E to F . Then the following conditions are equivalent:*

- (i) u is an isometric, surjective A -linear map;
- (ii) u is a unitary element of $\mathcal{L}(E, F)$.

Proposition 4 *With A, E, F as before, let w be a linear map from E to F . The following conditions are equivalent:*

- (i) w is an isometric A -linear map with complemented range;

(ii) $w \in \mathcal{L}(E, F)$ and $w^*w = 1_E$.

Let A, B be C^* -algebras and let E be a Hilbert B -module. Suppose that $\phi : A \longrightarrow \mathcal{L}(E)$ is a $*$ -homomorphism of C^* -algebras. We can also regard E as a left A -module, the action being given by

$$(a, y) \mapsto \phi(a)y \quad (a \in A, y \in E).$$

In this situation, we call E is a $A - B$ correspondence.

If \mathcal{H} is a Hilbert space then the algebraic (vector space) tensor product $\mathcal{H} \otimes_{\text{alg}} A$ (which is a right A -module, the module action being $(\xi \otimes a)b = \xi \otimes ab$ ($\xi \in \mathcal{H}; a, b \in A$)) has an A -valued inner-product given on simple tensors by

$$\langle \xi \otimes a \mid \eta \otimes b \rangle = \langle \xi \mid \eta \rangle a^*b \quad (\xi, \eta \in \mathcal{H}, a, b \in A).$$

It can be verified that this is a positive definite inner-product on $\mathcal{H} \otimes_{\text{alg}} A$ [13, p. 6]. Thus $\mathcal{H} \otimes_{\text{alg}} A$ is an inner-product A -module, we denote its completion by $\mathcal{H} \otimes A$. In the case where \mathcal{H} is a separable, infinite-dimensional Hilbert space, the Hilbert A -module $\mathcal{H} \otimes A$ is often denoted by \mathcal{H}_A . If E is a Hilbert A -module and $Z \subseteq E$ then we say that Z is a *generating set* for E if the closed submodule of E generated by Z is the whole of E . We say that E is countably generated if it has a countable generating set. We now state a theorem, known as Kasparov's stabilisation theorem. Intuitively, the idea of the theorem is that \mathcal{H}_A is big enough to absorb any countably generated Hilbert A -module; or alternatively that once a module reaches the size of \mathcal{H}_A it stabilises, cannot get any "bigger".

Theorem 5 [13, p. 60] *If A is a C^* -algebra and E is a countably generated Hilbert A -module then $E \oplus \mathcal{H}_A \approx \mathcal{H}_A$.*

We say that a closed submodule F of a Hilbert C^* -module E is *fully complemented* if F is complemented in E and $F^\perp \approx E$.

Corollary 6 *If E is a countably generated Hilbert A -module then E is unitarily equivalent to a fully complemented submodule of \mathcal{H}_A .*

Suppose that A, B are C^* -algebras, E is a Hilbert A -module and F is a Hilbert B -module. We want to define $E \otimes F$ as a Hilbert $(A \otimes B)$ -module. Start by forming the algebraic tensor product $E \otimes_{\text{alg}} F$ of the vector spaces E and F (over \mathbb{C}). This is a right module over $A \otimes_{\text{alg}} B$ (the module action being given by $(x \otimes y)(a \otimes b) = xa \otimes yb$). For x_1, x_2 in E and y_1, y_2 in F , we define

$$\langle x_1 \otimes y_1 \mid x_2 \otimes y_2 \rangle = \langle x_1 \mid x_2 \rangle \otimes \langle y_1 \mid y_2 \rangle.$$

This extends by linearity to an $(A \otimes_{\text{alg}} B)$ -valued sesquilinear form on $E \otimes_{\text{alg}} F$ which makes $E \otimes_{\text{alg}} F$ into a semi-inner-product $(A \otimes_{\text{alg}} B)$ -module over the pre- C^* -algebra $A \otimes_{\text{alg}} B$ [13, p. 34]. It can be shown that the semi-inner-product on $E \otimes_{\text{alg}} F$ is actually an inner-product by Kasparov's stabilisation theorem [13, p. 62]. The double completion process [13, p. 4] can be performed to conclude that the completion $E \otimes F$ of on $E \otimes_{\text{alg}} F$ is a Hilbert $(A \otimes B)$ -module. We call $E \otimes F$ the *exterior tensor product* of E and F .

Proposition 7 *With notations as in Section II. We have $\mathfrak{H} \approx L^2(X, \mu) \otimes E$.*

Proof. The unitary map $v : L^2(X, \mu) \otimes E \longrightarrow \mathfrak{H}$ is given by defining $v(F \otimes \xi)$ to be the map $x \mapsto F(x) \otimes \xi$ ($x \in X$), where $F \in L^2(X, \mu), \xi \in E$. It is straightforward to verify that v is a unitary.

V.2 Bijection

Here we write down an explicit bijection from \mathbb{N} to $\mathbb{N} \times \mathbb{N}$.

Proposition 8 *For $k \in \mathbb{N}, k > 1$ we have*

(i) *there is a unique integer $n_k \in \mathbb{N}$ such that $\frac{n_k(n_k+1)}{2} < k \leq \frac{(n_k+1)(n_k+2)}{2}$*

(ii) $t(k) := \begin{cases} (1, 1), & \text{for } k = 1; \\ \left(\frac{n_k(n_k+3)}{2} + 2 - k, k - \frac{n_k(n_k+1)}{2}\right), & \text{for } k > 1. \end{cases}$

is a one-to-one map from \mathbb{N} onto $\mathbb{N} \times \mathbb{N}$.

Proof. We observe that $\mathcal{I} := \{(\frac{n(n+1)}{2}, \frac{(n+1)(n+2)}{2}) : n \in \mathbb{N}\}$ is a family of non-intersecting intervals in the real line whose union is the interval $(1, \infty)$. So any real number $x > 1$ can belong to a unique interval in \mathcal{I} . In particular, this is true for any $k \in \mathbb{N}, k > 1$. This proves (i).

To prove t is one-to-one we show that $t(k) = t(k')$ implies $k = k'$. If $n_k = n'_k$ then it is clear from the expression for $t(k)$ that $t(k) = t(k')$ implies $k = k'$. To settle the other possibility, let $n_k \neq n'_k$ and $t(k) = t(k')$. By hypothesis we have $(\frac{n_k(n_k+3)}{2} + 2 - k, k - \frac{n_k(n_k+1)}{2}) = (\frac{n_{k'}(n_{k'}+3)}{2} + 2 - k', k' - \frac{n_{k'}(n_{k'}+1)}{2})$. Equating the first component of the ordered pairs we have $k' - k = \frac{1}{2}(n_{k'} - n_k)(n_{k'} + n_k + 3)$. Similarly, we obtain $k' - k = \frac{1}{2}(n_{k'} - n_k)(n_{k'} + n_k + 1)$ from the second component of the ordered pair. Equating the expressions for $k' - k$ we get $n_k = n_{k'}$. Therefore, we are reduced to the previous case and hence $k = k'$.

Finally, we show that t is onto. Given any $(p, q) \in \mathbb{N} \times \mathbb{N}$ we set $n_k = p + q - 2$ and $k = \frac{n_k(n_k+3)}{2} + 2 - p = \frac{n_k(n_k+1)}{2} + q$. We see that $t(k) = (p, q)$. This completes the proof.

References

- [1] S.L. Adler and A.C. Millard, *Coherent states in quaternionic quantum mechanics*, J. Math. Phys. **38**, 2117-2126 (1996).
- [2] S.L. Adler *Quaternionic Quantum Mechanics and Quantum Fields*, Oxford University Press (1995).
- [3] S.T. Ali, J.-P. Antoine and J.-P. Gazeau, *Coherent States, Wavelets and their Generalizations*, Springer-Verlag, New York (1999).
- [4] S.T. Ali and F. Bagarello, *Some physical appearances of vector coherent states and coherent states related to degenerate Hamiltonians*, J. Math. Phys. **46**, 053518-1 – 053518-28 (2005), online version.
- [5] S.T. Ali, M. Engliš and J.-P. Gazeau, *Vector coherent states from Plancherel's theorem, Clifford algebras and matrix domains*, J. Phys. **A37**, 6067-6089 (2004).
- [6] S.T. Ali, and M. Engliš, *Berezin-Toeplitz quantization over matrix domains*, in *Contributions in Mathematical Physics: A Tribute to Gérard G. Emch*, Eds. S.T. Ali and K.B. Sinha, Hindustan Book Agency, New Delhi, India (2007).
- [7] J. A. Ball, A. Biswas, Q. Fang, and S. ter Horst, *Multivariable generalizations of the Schur class: Positive kernel characterization and transfer function realization*, T. Ando (ed.) et al., Recent advances in operator theory and applications. Proceedings of the international workshop on operator theory and applications (IWOTA), Seoul, Korea, July 31–August 3, 2006. Basel: Birkhuser. Operator Theory: Advances and Applications 187, 17-79 (2009).
- [8] S.D. Barreto, B.V.R. Bhat, V. Liebscher and M. Skeide, *Type I product systems of Hilbert modules*, J. Funct. Anal. **212**, 121-181 (2004).
- [9] F. M. Brückler, *Tensor products of C^* -algebras, operator spaces and Hilbert C^* -modules*, Math. Commun. **4**, 257-268 (1999).
- [10] J. Cuntz, *Simple C^* -algebras generated by isometries*, Comm. Math. Phys. **57**, 173-185 (1977).
- [11] J.-P. Gazeau, *Coherent States in quantum Physics*, WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim (2009).

- [12] J. Ginibre, *Statistical ensembles of complex, quaternion and real matrices*, J. Math. Phys., **6**, 440-449 (1965).
- [13] E.C. Lance, *Hilbert C^* -Modules*, A toolkit for operator algebraists, Lond. Math. Soc. Lec. Notes Series. 210, Cambridge University Press, Cambridge (1995).
- [14] A.M. Perelomov, *Generalized Coherent States and their applications*, Springer-Verlag, Berlin (1986).
- [15] K. Thirulogasanthar and S.T. Ali, *A class of vector coherent states defined over matrix domains*, J. Math. Phys. **44**, 5070-5083 (2003).
- [16] K. Thirulogasanthar, G. Honnouvo and A. Krzyzak, *Coherent states and Hermite polynomials on quaternionic Hilbert spaces*, Concordia University preprint (2010).